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THE STRESS DISTRIBUTION IN AN INFINITE ANISOTROPIC PLATE WITH CO-LINEAR CRACKS

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Abstract—A general solution of the plane problem of a finite number of co-linear cracks in an anisotropic material is presented. The solution is obtained by reducing the problem to four very simple Riemann–Hilbert problems. From the solution it is concluded that if the loads acting on the cracks have the resultant zero for each of the cracks, then the normal and shear stresses created on the line of the cracks are independent of the elastic constants. Expressions for the stress intensity factors are derived, and some examples are presented.

INTRODUCTION

The calculation of stresses in isotropic elastic plates by means of complex potentials has been well established for some time[1]. One of the standard problems is the calculation of stresses in an infinite plate with an arbitrary finite number of co-linear cracks. The solution includes arbitrary surface loads and a constant stress state at infinity. The purpose of this paper is to extend this rigorous method to anisotropic elastic materials. Although some results have been obtained[2–4] a general method seems to be wanted.

Some general results will be taken from [5]. Hooke's law will be used in the form

$$\begin{aligned} \epsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\sigma_{xy} \\ \epsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\sigma_{xy} \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\sigma_{xy} \end{aligned} \qquad \text{(1a–c)}$$

for plane stress. For plane strain the elastic constants a_{ij} will be exchanged with

$$a^*_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad i, j = 1, 2, 6. \qquad (2)$$

The equilibrium equations are satisfied by stresses given by Airy's stress function U.

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \qquad \text{(3a–c)}$$

By using (3) and (1) we can write the compatibility equation

$$a_{22}\frac{\partial^4 U}{\partial x^4} - 2a_{26}\frac{\partial^4 U}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 U}{\partial x \partial y^3} + a_{11}\frac{\partial^4 U}{\partial y^4} = 0. \qquad (4)$$

We define the complex constants s_1 and s_2 as roots of the equation

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \qquad (5)$$

and write

$$s_1 = \alpha_1 + i\beta_1, \quad s_2 = \alpha_2 + i\beta_2 \tag{6a, b}$$

where α_j and β_j are real constants, and the β_j's are positive. The following is restricted to the cases where the roots s_1 and s_2 are distinct. If they are not the problem may be treated as the corresponding isotropic case. We then have the general solution to (4)

$$U(x, y) = 2Re[U_1(z_1) + U_2(z_2)] \tag{7}$$

where $U_1(z_1)$ and $U_2(z_2)$ are arbitrary functions of the complex variables $z_j = x + s_j y$. Four complex functions are defined by

$$\phi_j(z_j) = \frac{dU_j}{dz_j}, \quad \Phi_j(z_j) = \frac{d\phi_j}{dz_j} \qquad j = 1, 2. \tag{8a, b}$$

The stresses can then be expressed as

$$\begin{aligned} \sigma_x &= 2Re[s_1^2\Phi_1(z_1) + s_2^2\Phi_2(z_2)] \\ \sigma_y &= 2Re[\Phi_1(z_1) + \Phi_2(z_2)] \\ \sigma_{xy} &= -2Re[s_1\Phi_1(z_1) + s_2\Phi_2(z_2)] \end{aligned} \tag{9a–c}$$

and by integration the displacements are found to be

$$\begin{aligned} u &= 2Re[p_1\phi_1(z_1) + p_2\phi_2(z_2)] \\ v &= 2Re[q_1\phi_1(z_1) + q_2\phi_2(z_2)]. \end{aligned} \tag{10a, b}$$

p_j and q_j are complex constants defined by

$$\begin{aligned} p_j &= a_{11}s_j^2 + a_{12} - a_{16}s_j \\ q_j &= (a_{12}s_j^2 + a_{22} - a_{26}s_j)/s_j \qquad j = 1, 2 \end{aligned} \tag{11a, b}$$

THE RIEMANN-HILBERT PROBLEM

Let the same linearly anisotropic medium occupy the upper half plane, S^+, and the lower half plane, S^- (Fig. 1). The two half planes are bonded along a part of the real axis. On the rest of the real axis, L, a finite number of cracks, $L_k = [a_k, b_k]$, $k = 1, 2, \ldots, n$, are formed. The cracks are of finite length. The special case of infinite cracks is illustrated in Example 2. The stresses on the crack surfaces and a homogeneous stress state at infinity are known.

In order to formulate the problem as a Riemann–Hilbert problem we introduce four complex functions by

$$\begin{aligned} 2\Psi_j(z_j) &= \Phi_j(z_j) + \bar{\Phi}_j(z_j) \\ 2\Omega_j(z_j) &= \Phi_j(z_j) - \bar{\Phi}_j(z_j) \qquad j = 1, 2 \end{aligned} \tag{12a, b}$$

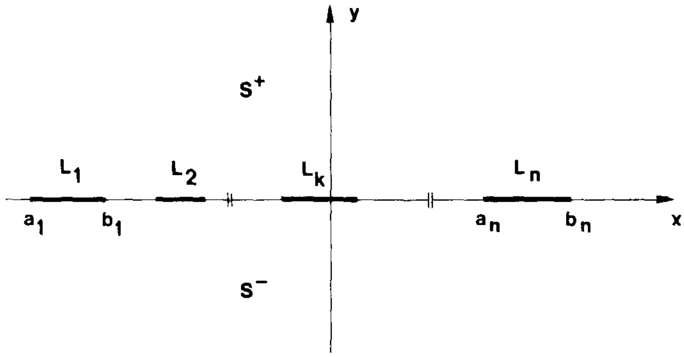

Fig. 1.

where we have used the notation

$$\bar{\Phi}_j(z_j) = \overline{\Phi_j(\bar{z}_j)}. \tag{13}$$

The functions $\Phi_j(z_j)$ are assumed to be analytic outside L. It is then easy to prove that the same is the case for $\Psi_j(z_j)$ and $\Omega_j(z_j)$[1, §76]. Using (12) and (13) we obtain

$$\begin{aligned}\Phi_j(z_j) &= \Psi_j(z_j) + \Omega_j(z_j)\\ \overline{\Phi_j(z_j)} &= \Psi_j(\bar{z}_j) - \Omega_j(\bar{z}_j) \qquad j = 1, 2.\end{aligned} \tag{14a, b}$$

The stresses σ_y and σ_{xy} may now be expressed as—(9), (14)

$$\begin{aligned}\sigma_y &= \sum_j [\Psi_j(z_j) + \Omega_j(z_j) + \Psi_j(\bar{z}_j) - \Omega_j(\bar{z}_j)]\\ \sigma_{xy} &= -\sum_j [s_j\Psi_j(z_j) + s_j\Omega_j(z_j) + \bar{s}_j\Psi_j(\bar{z}_j) - \bar{s}_j\Omega_j(\bar{z}_j)].\end{aligned} \tag{15a, b}$$

By taking upper and lower limits in (15) on L the following expressions are obtained

$$\begin{aligned}\sigma_y^+ &= [\Psi_1 + \Psi_2 + \Omega_1 + \Omega_2]^+ + [\Psi_1 + \Psi_2 - \Omega_1 - \Omega_2]^-\\ \sigma_y^- &= [\Psi_1 + \Psi_2 - \Omega_1 - \Omega_2]^+ + [\Psi_1 + \Psi_2 + \Omega_1 + \Omega_2]^-\\ \sigma_{xy}^+ &= -[s_1\Psi_1 + s_2\Psi_2 + s_1\Omega_1 + s_2\Omega_2]^+ - [\bar{s}_1\Psi_1 + \bar{s}_2\Psi_2 - \bar{s}_1\Omega_1 - \bar{s}_2\Omega_2]^-\\ \sigma_{xy}^- &= -[\bar{s}_1\Psi_1 + \bar{s}_2\Psi_2 - \bar{s}_1\Omega_1 - \bar{s}_2\Omega_2]^+ - [s_1\Psi_1 + s_2\Psi_2 + s_1\Omega_1 + s_2\Omega_2]^-.\end{aligned} \tag{16a–d}$$

The loads on the crack surfaces define the four functions

$$\begin{aligned}f(t) &= (\sigma_y^+ + \sigma_y^-)/2, \quad g(t) = (\sigma_y^+ - \sigma_y^-)/2\\ h(t) &= (\sigma_{xy}^+ + \sigma_{xy}^-)/2, \quad k(t) = (\sigma_{xy}^+ - \sigma_{xy}^-)/2 \qquad t \in L.\end{aligned} \tag{17a–d}$$

By addition and subtraction the equations (16) become

$$[\Psi_1 + \Psi_2]^+ + [\Psi_1 + \Psi_2]^- = f(t)$$

$$[\Omega_1+\Omega_2]^+-[\Omega_1+\Omega_2]^-=g(t)$$

$$[\alpha_1\Psi_1+\alpha_2\Psi_2+i\beta_1\Omega_1+i\beta_2\Omega_2]^++[\alpha_1\Psi_1+\alpha_2\Psi_2+i\beta_1\Omega_1+i\beta_2\Omega_2]^-=-h(t)$$

$$[i\beta_1\Psi_1+i\beta_2\Psi_2+\alpha_1\Omega_1+\alpha_2\Omega_2]^+-[i\beta_1\Psi_1+i\beta_2\Psi_2+\alpha_1\Omega_1+\alpha_2\Omega_2]^-=-k(t) \qquad t\in L. \quad \text{(18a–d)}$$

This is the standard formulation of the Riemann–Hilbert problem, and the solutions can be written down directly[1].

$$\Psi_1(z)+\Psi_2(z)=\frac{1}{2\pi iX(z)}\int_L\frac{X^+(t)f(t)}{t-z}\,\mathrm{d}t+\frac{P(z)}{X(z)}$$

$$\Omega_1(z)+\Omega_2(z)=\frac{1}{2\pi i}\int_L\frac{g(t)}{t-z}\,\mathrm{d}t+iC$$

$$\alpha_1\Psi_1(z)+i\beta_1\Omega_1(z)+\alpha_2\Psi_2(z)+i\beta_2\Omega_2(z)$$
$$=-\frac{1}{2\pi iX(z)}\int_L\frac{X^+(t)h(t)}{t-z}\,\mathrm{d}t-\frac{Q(z)}{X(z)}$$

$$i\beta_1\Psi_1(z)+\alpha_1\Omega_1(z)+i\beta_2\Psi_2(z)+\alpha_2\Omega_2(z)$$
$$=-\frac{1}{2\pi i}\int_L\frac{k(t)}{t-z}\,\mathrm{d}t-iD \qquad \text{(19a–d)}$$

where

$$X(z)=\prod_{k=1}^{n}\sqrt{(z-a_k)(z-b_k)} \qquad (20)$$

and we select the branch for which $X(z)/z^n\to 1$ for $z\to\infty$. $P(z)$ and $Q(z)$ are polynomials defined by

$$P(z)=\sum_{k=0}^{n}A_kz^k,\quad Q(z)=\sum_{k=0}^{n}B_kz^k. \qquad \text{(21a, b)}$$

From (12) we obtain the relations

$$\bar{\Psi}_j(z)=\Psi_j(z),\quad \bar{\Omega}_j(z)=-\Omega_j(z) \qquad j=1,2 \qquad \text{(22a, b)}$$

from which it follows that the constants C, D, A_k and B_k, $k=0,1,\ldots,n$, are real. The original complex stress functions $\Phi_1(z)$ and $\Phi_2(z)$ are found by adding (19a) to (19b) and (19c) to (19d).

$$\Phi_1(z)+\Phi_2(z)=\frac{1}{2\pi iX(z)}\int_L\frac{X^+(t)f(t)}{t-z}\,\mathrm{d}t+\frac{P(z)}{X(z)}$$
$$+\frac{1}{2\pi i}\int_L\frac{g(t)}{t-z}\,\mathrm{d}t+iC$$

$$s_1\Phi_1(z)+s_2\Phi_2(z)=-\frac{1}{2\pi iX(z)}\int_L\frac{X^+(t)h(t)}{t-z}\,\mathrm{d}t-\frac{Q(z)}{X(z)}$$
$$-\frac{1}{2\pi i}\int_L\frac{k(t)}{t-z}\,\mathrm{d}t-iD. \qquad \text{(23a, b)}$$

The stress state at infinity is determined by σ_x^∞, σ_y^∞ and σ_{xy}^∞. By making use of (23) and (9) we find

$$A_n = \sigma_y^\infty/2, \quad B_n = \sigma_{xy}^\infty/2 \tag{24a, b}$$

and a relation between C and D,

$$(\alpha_1\beta_2 + \alpha_2\beta_1)C + (\beta_1 + \beta_2)D = \sigma_x^\infty/2 + (\alpha_1\alpha_2 - \beta_1\beta_2)\sigma_y^\infty/2 + (\alpha_1 + \alpha_2)\sigma_{xy}^\infty/2. \tag{25}$$

In the general case C or D may be chosen arbitrarily.

The final solution of the problem is expressed by

$$(s_2 - s_1)\Phi_1(z_1) = \frac{1}{2\pi i X(z_1)}\int_L \frac{X^+(t)[s_2 f(t) + h(t)]}{t - z_1}\,dt + \frac{1}{2\pi i}\int_L \frac{s_2 g(t) + k(t)}{t - z_1}\,dt + \frac{s_2 P(z_1) + Q(z_1)}{X(z_1)} + i(s_2 C + D)$$

$$(s_1 - s_2)\Phi_2(z_2) = \frac{1}{2\pi i X(z_2)}\int_L \frac{X^+(t)[s_1 f(t) + h(t)]}{t - z_2}\,dt + \frac{1}{2\pi i}\int_L \frac{s_1 g(t) + k(t)}{t - z_2}\,dt + \frac{s_1 P(z_2) + Q(z_2)}{X(z_2)} + i(s_1 C + D). \tag{26a, b}$$

The solution still includes the $2n$ arbitrary constants A_k and B_k, $k = 0, 1, \ldots, n-1$. They are determined by constraints on the displacements.

$$\int_{a_k}^{b_k}\left[\frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}\right] dx = 0$$

$$\int_{a_k}^{b_k}\left[\frac{\partial v^+}{\partial x} - \frac{\partial v^-}{\partial x}\right] dx = 0 \qquad k = 1, 2, \ldots, n \tag{27a, b}$$

(27) is written by means of Φ_1 and Φ_2.

$$Re\int_{a_k}^{b_k}\{p_1[\Phi_1^+(x) - \Phi_1^-(x)] + p_2[\Phi_2^+(x) - \Phi_2^-(x)]\}\,dx = 0$$

$$Re\int_{a_k}^{b_k}\{q_1[\Phi_1^+(x) - \Phi_1^-(x)] + q_2[\Phi_2^+(x) - \Phi_2^-(x)]\}\,dx = 0 \qquad k = 1, 2, \ldots, n. \tag{28a, b}$$

Some general features of the solution may be revealed by a closer analysis of the equations (28). The equations (28a) are written as

$$Re\left\{\frac{p_1 s_2 - p_2 s_1}{s_2 - s_1}\int_{a_k}^{b_k}\left[\frac{1}{\pi i X^+(x)}\int_L \frac{X^+(t) f(t)}{t - x}\,dt + g(x) + \frac{2P(x)}{X^+(x)}\right] dx + \frac{p_1 - p_2}{s_2 - s_1}\int_{a_k}^{b_k}\left[\frac{1}{\pi i X^+(x)}\int_L \frac{X^+(t) h(t)}{t - x}\,dt + k(x) + \frac{2Q(x)}{X^+(x)}\right] dx\right\} = 0 \qquad k = 1, 2, \ldots, n. \tag{29}$$

The equations (28b) are derived from (29) by exchanging p_j with q_j, $j = 1, 2$. The elastic constants are evaluated by means of (11).

$$\frac{p_1 s_2 - p_2 s_1}{s_2 - s_1} = a_{12} - a_{11} s_1 s_2$$

$$\frac{p_1 - p_2}{s_2 - s_1} = a_{16} - a_{11}(s_1 + s_2)$$

$$\frac{q_1 s_2 - q_2 s_1}{s_2 - s_1} = a_{22} \frac{s_1 + s_2}{s_1 s_2} - a_{26}$$

$$\frac{q_1 - q_2}{s_2 - s_1} = a_{22} \frac{1}{s_1 s_2} - a_{12}. \tag{30a–d}$$

The equations (29) are rewritten as

$$\mathrm{Im}\left[\frac{p_1 s_2 - p_2 s_1}{s_2 - s_1}\right] \int_{a_k}^{b_k} \left[\frac{1}{\pi i X^+(x)} \int_L \frac{X^+(t) f(t)}{t - x} \mathrm{d}t + \frac{2P(x)}{X^+(x)}\right] \mathrm{d}x$$

$$+\mathrm{Im}\left[\frac{p_1 - p_2}{s_2 - s_1}\right] \int_{a_k}^{b_k} \left[\frac{1}{\pi i X^+(x)} \int_L \frac{X^+(t) h(t)}{t - x} dt + \frac{2Q(x)}{X^+(x)}\right] \mathrm{d}x$$

$$= i\mathrm{Re}\left[\frac{p_1 s_2 - p_2 s_1}{s_2 - s_1}\right] \int_{a_k}^{b_k} g(x)\, \mathrm{d}x + i\mathrm{Re}\left[\frac{p_1 - p_2}{s_2 - s_1}\right] \int_{a_k}^{b_k} k(x)\, \mathrm{d}x$$

$$k = 1, 2, \ldots, n. \tag{31}$$

Analogous equations containing q_1 and q_2 are obtained from (27b).

Let us consider loads for which

$$\int_{a_k}^{b_k} g(x)\, \mathrm{d}x = 0, \qquad \int_{a_k}^{b_k} k(x)\, \mathrm{d}x = 0 \qquad k = 1, \ldots, n. \tag{32a, b}$$

The system (31) and the analogous equations then become homogeneous and may be reduced to

$$\int_{a_k}^{b_k} \left[\frac{1}{\pi i X^+(x)} \int_L \frac{X^+(t) f(t)}{t - x} \mathrm{d}t + \frac{2P(x)}{X^+(x)}\right] \mathrm{d}x = 0$$

$$\int_{a_k}^{b_k} \left[\frac{1}{\pi i X^+(x)} \int_L \frac{X^+(t) h(t)}{t - x} \mathrm{d}t + \frac{2Q(x)}{X^+(x)}\right] \mathrm{d}x = 0 \qquad k = 1, \ldots, n. \tag{33a, b}$$

This means that for loads of the type (32) $P(z)$ and $Q(z)$ are independent of the elastic constants. From (23) and (9) we can then conclude that in this case the stresses $\sigma_y(x, 0)$ and $\sigma_{xy}(x, 0)$ are independent of the elastic constants. A similar theorem holds for the isotropic case [1, §120] from which it follows that the stresses $\sigma_y(x, 0)$ and $\sigma_{xy}(x, 0)$ generally may be obtained from the isotropic case when the conditions (32) are fulfilled. Under the conditions (32) it is furthermore clear that normal surface stresses will not produce shear stresses along the x-axis and conversely.

STRESSES AROUND THE CRACK TIPS

The stress distributions around the crack tips depend upon the asymptotic behavior of the complex stress functions Φ_1 and Φ_2. From the general solution (26) we find the asymptotic behavior in the neighborhood of the crack tip $z = b_m$.

$$\Phi_j(z_j) = \frac{F_j}{2\sqrt{2(z_j - b_m)}} + 0(1) \qquad j = 1, 2. \tag{34}$$

F_1 and F_2 are complex constants determined by

$$F_j = 2 \lim_{z_j \to b_m} \sqrt{2(z_j - b_m)}\Phi_j(z_j) \qquad j = 1, 2. \tag{35}$$

The usual square root singularity is present, and the stress intensity factors are introduced by the conventional definitions

$$\sigma_y(x, 0) = \frac{k_1}{\sqrt{2(x - b_m)}} + 0(1)$$

$$\sigma_{xy}(x, 0) = \frac{k_2}{\sqrt{2(x - b_m)}} + 0(1). \tag{36a, b}$$

From (9) and (23) we obtain the relations

$$Re[F_1 + F_2] = k_1, \quad \text{Im}[F_1 + F_2] = 0$$

$$Re[s_1F_1 + s_2F_2 = -k_2, \quad \text{Im}[s_1F_1 + s_2F_2] = 0. \tag{37a–d}$$

This is equivalent to

$$k_1 = 2 \lim_{x \to b_m} \sqrt{2(x - b_m)}[\Phi_1(x) + \Phi_2(x)]$$

$$k_2 = 2 \lim_{x \to b_m} \sqrt{2(x - b_m)}[s_1\Phi_1(x) + s_2\Phi_2(x)] \tag{38a, b}$$

and leads to the asymptotic stress field

$$\sigma_x = \frac{1}{\sqrt{2r}} Re\left[\frac{s_1^2F_1}{\sqrt{\cos\theta + s_1 \sin\theta}} + \frac{s_2^2F_2}{\sqrt{\cos\theta + s_2 \sin\theta}}\right] + 0(1)$$

$$\sigma_y = \frac{1}{\sqrt{2r}} Re\left[\frac{F_1}{\sqrt{\cos\theta + s_1 \sin\theta}} + \frac{F_2}{\sqrt{\cos\theta + s_2 \sin\theta}}\right] + 0(1)$$

$$\sigma_{xy} = \frac{-1}{\sqrt{2r}} Re\left[\frac{s_1F_1}{\sqrt{\cos\theta + s_1 \sin\theta}} + \frac{s_2F_2}{\sqrt{\cos\theta + s_2 \sin\theta}}\right] + 0(1) \tag{39a–c}$$

where we have used

$$z - b_m = re^{i\theta} = r(\cos\theta + i \sin\theta) \tag{40}$$

and

$$F_1 = \frac{s_2 k_1 + k_2}{s_2 - s_2}, \quad F_2 = \frac{s_1 k_1 + k_2}{s_1 - s_2}. \tag{41a, b}$$

We note that the asymptotic stress field is determined by two real parameters, k_1 and k_2, and the elastic properties of the material. The ratio of the normal stresses ahead of the crack is given by

$$\lim_{x \to b_m} \frac{\sigma_x}{\sigma_y} = -Re\left[s_1 s_2 + (s_1 + s_2)\frac{k_2}{k_1} \right]. \tag{42}$$

In the case of general anisotropy this ratio depends on the loading through the parameter k_2/k_1. For orthotropic materials with axes aligned parallel and perpendicular to the crack the ratio is a material constant because $Re[s_1 + s_2] = 0$.

EXAMPLE 1

A single crack extending from $-a$ to a is considered (Fig. 2). Constant self-equilibrating loads are applied to the crack surfaces.

$$\begin{aligned} f(t) &= -p, \quad g(t) = 0 \\ h(t) &= -q, \quad k(t) = 0 \qquad -a < t < a. \end{aligned} \tag{43}$$

The stresses vanish at infinity, and the solution is then given by

$$\begin{aligned} (s_2 - s_1)\Phi_1(z_1) &= -\frac{s_2 p + q}{2\pi\sqrt{z_1^2 - a^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{t - z_1} \mathrm{d}t + \frac{s_2 A_0 + B_0}{\sqrt{z_1^2 - a^2}} \\ (s_1 - s_2)\Phi_2(z_2) &= -\frac{s_1 p + q}{2\pi\sqrt{z_2^2 - a^2}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{t - z_2} \mathrm{d}t + \frac{s_1 A_0 + B_0}{\sqrt{z_2^2 - a^2}}. \end{aligned} \tag{44a, b}$$

By performing the integrations in (33) we get $A_0 = B_0 = 0$ and thereby

$$(s_2 - s_1)\Phi_1(z_1) = \frac{s_2 p + q}{2\sqrt{z_1^2 - a^2}} \left[z_1 - \sqrt{z_1^2 - a^2} \right]$$

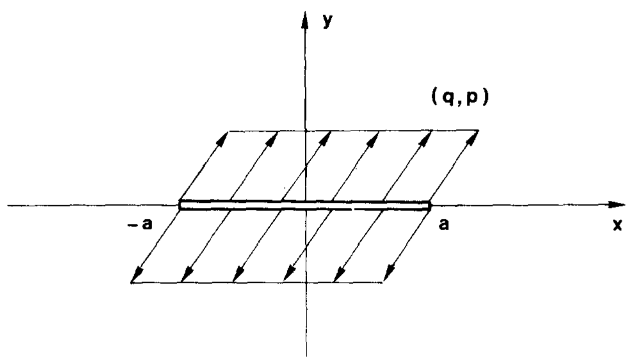

Fig. 2.

$$(s_1 - s_2)\Phi_2(z_2) = \frac{s_1 p + q}{2\sqrt{z_2^2 - a^2}}[z_2 - \sqrt{z_2^2 - a^2}]. \quad (45a, b)$$

The stress intensity factors follow from (38).

$$k_1 = p\sqrt{a}, \qquad k_2 = q\sqrt{a} \quad (46a, b)$$

(46) could have been obtained directly from the equality with the corresponding results from the isotropic case.

EXAMPLE 2

In this example we consider two cracks extending from a and $-a$ to plus and minus infinity (Fig. 3). The loading consists of the transmission of a finite force (q, p) and a moment m across the x-axis. In this problem we use the fundamental function

$$X(z) = \sqrt{a^2 - z^2} \quad (47)$$

where we use the branch for which $X(z)/z \to i$ for $z \to \infty$ in S^-. Hereby we obtain the solution

$$(s_2 - s_1)\Phi_1(z_1) = \frac{s_2 P(z_1) + Q(z_1)}{\sqrt{a^2 - z_1^2}} + i(s_2 C + D)$$

$$(s_1 - s_2)\Phi_2(z_2) = \frac{s_1 P(z_2) + Q(z_2)}{\sqrt{a^2 - z_2^2}} + i(s_1 C + D). \quad (48a, b)$$

At infinity the stresses are zero, and use of (9a) yields

$$Re[is_1 s_2 A_1 + i(s_1 + s_2)B_1 - is_1 s_2 C - i(s_1 + s_2)D] = 0 \quad \text{for } z \in S^-$$

$$Re[-is_1 s_2 A_1 - i(s_1 + s_2)B_1 - is_1 s_2 C - i(s_1 + s_2)D] = 0 \quad \text{for } z \in S^+. \quad (49a, b)$$

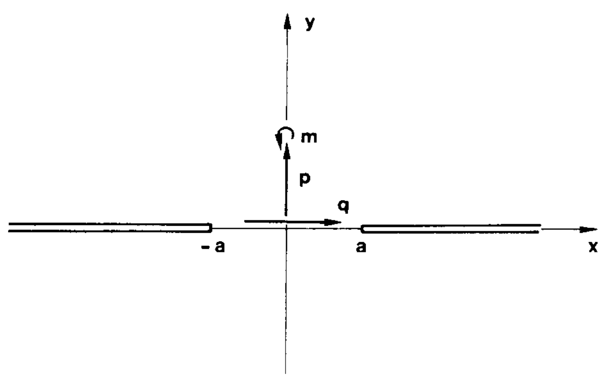

Fig. 3.

We may use $C = D = 0$ and get

$$B_1 = -\frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\beta_1 + \beta_2} A_1. \tag{50}$$

The constants A_1, A_0 and B_0 are determined from statical conditions

$$2A_1 \int_{-a}^{a} \frac{x^2}{\sqrt{a^2 - x^2}} \mathrm{d}x = m$$

$$2A_0 \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = p$$

$$2B_0 \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = q. \tag{51a–c}$$

The solution is determined by (48), (50) and the constants

$$A_1 = \frac{m}{\pi a^2}, \quad A_0 = \frac{p}{2\pi}, \quad B_0 = \frac{q}{2\pi}. \tag{52a–c}$$

The stress intensity factors at $z = a$ are easily found from (38) and (48).

$$k_1 = \frac{1}{\pi\sqrt{a}}\left[2\frac{m}{a} + p\right]$$

$$k_2 = \frac{1}{\pi\sqrt{a}}\left[-2\frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\beta_1 + \beta_2}\frac{m}{a} + q\right]. \tag{53a, b}$$

For orthotropic materials the elastic constant in (53b) is zero, and the results are then equal to results obtained for the isotropic case[6].

EXAMPLE 3

Let us for simplicity consider a single crack extending from $-a$ to a in an orthotropic material.

$$a_{16} = 0, \quad a_{26} = 0 \tag{54a, b}$$

$$Re[s_1 + s_2] = 0, \quad \mathrm{Im}[s_1 s_2] = 0. \tag{55a, b}$$

The stresses vanish at infinity, and the surface load is a force (q, p) acting on the upper surface at $z = x_0$ (Fig. 4).

$$f(t) = g(t) = -p\delta(t - x_0)/2$$

$$h(t) = k(t) = -q\delta(t - x_0)/2. \tag{56a–d}$$

The solution is (26), (20)

$$(s_2 - s_1)\Phi_1(z_1) = \frac{1}{4\pi X(z_1)}\frac{\sqrt{a^2 - x_0^2}}{z_1 - x_0}[s_2 p + q] + \frac{1}{4\pi i}\frac{1}{z_1 - x_0}[s_2 p + q] + \frac{s_2 A_0 + B_0}{X(z_1)}$$

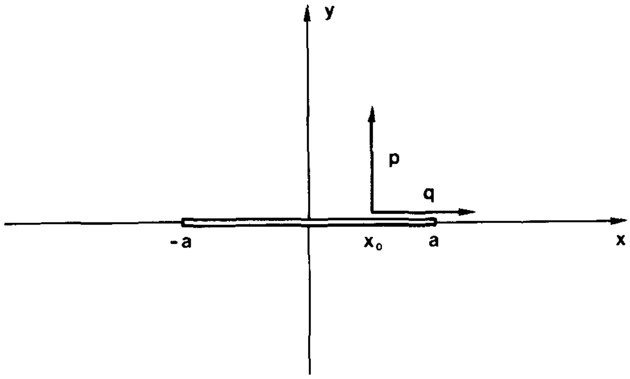

Fig. 4.

$$(s_1 - s_2)\Phi_2(z_2) = \frac{1}{4\pi X(z_2)}\frac{\sqrt{a^2 - x_0^2}}{z_2 - x_0}[s_1 p + q] + \frac{1}{4\pi i}\frac{1}{z_2 - x_0}[s_1 p + q] + \frac{s_1 A_0 + B_0}{X(z_2)}. \tag{57a, b}$$

A_0 and B_0 are determined from (31) and the corresponding equation containing q_j. We use that for an orthotropic material

$$\mathrm{Im}\left[\frac{p_1 s_2 - p_2 s_1}{s_2 - s_1}\right] = 0, \quad Re\left[\frac{p_1 - p_2}{s_2 - s_1}\right] = 0$$
$$Re\left[\frac{q_1 s_2 - q_2 s_1}{s_2 - s_1}\right] = 0, \quad \mathrm{Im}\left[\frac{q_1 - q_2}{s_2 - s_1}\right] = 0. \tag{58a–d}$$

A_0 and B_0 are then determined from the two independent equations

$$\int_{-a}^{a}\left[\frac{1}{2\pi X^+(x)}\frac{\sqrt{a^2 - x_0^2}}{x - x_0}p + \frac{2A_0}{X^+(x)}\right]\mathrm{d}x = \frac{a_{22} - a_{12}s_1 s_2}{2a_{22}(s_1 + s_2)}q$$
$$\int_{-a}^{a}\left[\frac{1}{2\pi X^+(x)}\frac{\sqrt{a^2 - x_0^2}}{x - x_0}q + \frac{2B_0}{X^+(x)}\right]\mathrm{d}x = -\frac{a_{12} - a_{11}s_1 s_2}{2a_{11}(s_1 + s_2)}p. \tag{59a, b}$$

By performing the integrations we obtain

$$A_0 = i\frac{a_{22} - a_{12}s_1 s_2}{4\pi a_{22}(s_1 + s_2)}q, \quad B_0 = -i\frac{a_{12} - a_{11}s_1 s_2}{4\pi a_{11}(s_1 + s_2)}p. \tag{60a, b}$$

Application of (38) leads to the stress intensity factors at $z = a$.

$$k_1 = \frac{p}{2\pi\sqrt{a}}\sqrt{\frac{a + x_0}{a - x_0}} + \frac{q}{2\pi\sqrt{a}}i\frac{a_{22} - a_{12}s_1 s_2}{a_{22}(s_1 + s_2)}$$
$$k_2 = \frac{q}{2\pi\sqrt{a}}\sqrt{\frac{a + x_0}{a - x_0}} - \frac{p}{2\pi\sqrt{a}}i\frac{a_{12} - a_{11}s_1 s_2}{a_{11}(s_1 + s_2)}. \tag{61a, b}$$

For an isotropic material

$$i\frac{a_{22}-a_{12}s_1s_2}{a_{22}(s_1+s_2)}=i\frac{a_{12}-a_{11}s_1s_2}{a_{11}(s_1+s_2)}=\frac{1-\nu}{2} \tag{62a, b}$$

where ν is Poisson's ratio. The stress intensity factors (61) then become identical to the factors for the isotropic case as given in [6].

The results (61) are different from similar results given in [3] for $q = 0$. In [3] k_1 depends on p and the elastic constants. This is in contradiction to a general conclusion following from (31) and (58) for orthotropic materials. For orthotropic materials with axes aligned along the x- and y-axis k_1's dependence on the vertical loads and k_2's dependence on the horizontal loads will not include the elastic constants of the material.

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